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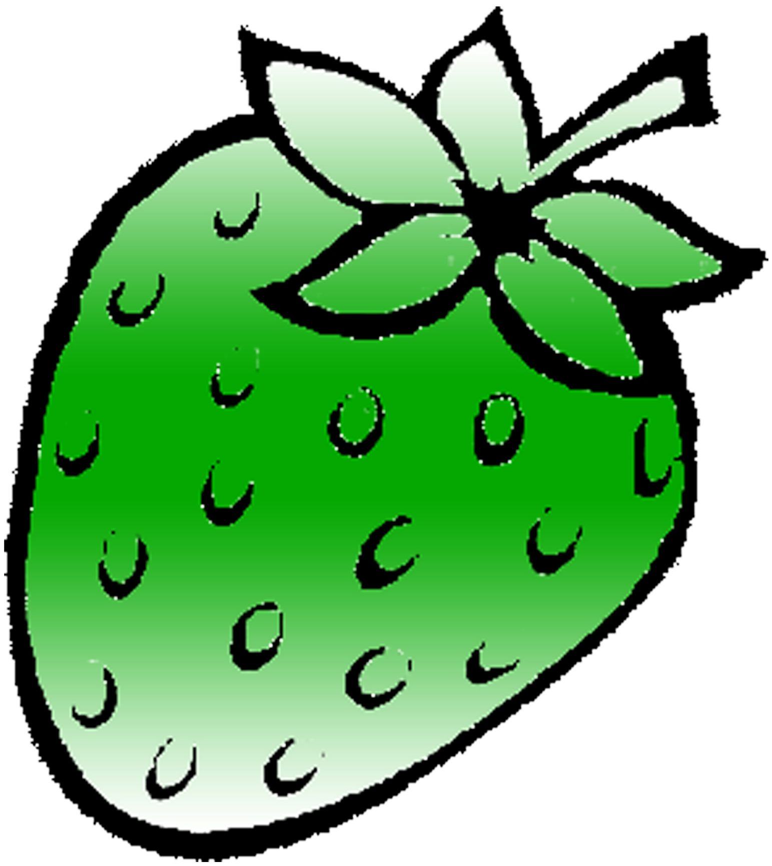

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SVKM's NMIMS

Mukesh Patel School of Technology Management and Engineering, Vile Parle, Mumbai- 400056

Tutorial Manual

Academic Year-2013-14

Program: B. Tech / MBA.Tech (All programmes)

Semester – I

Course: - Engineering Mathematics I

SVKM'S NMIMS

Mukesh Patel School of Technology Management & Engineering

First Year

Course: Engineering Mathematics I

Course Objectives:

1. To provide an understanding of Vector Algebra, Single variable and Multivariable Calculus and their applications.
2. To give an introduction to complex numbers and functions on complex numbers

Course Outcomes:

After the successful completion of this course, the student will be able to

1. Recall basics of Complex Numbers, DeMoivre's theorem, Mean Value Theorems and Vector triple products, Locate roots of complex numbers.
2. Discuss concept of Curves in space, gradient, Directional derivative, Curl and Divergence,
3. Express $\sin n\theta$, $\cos n\theta$ in powers of $\sin\theta$ and $\cos\theta$ and $\sin^n\theta$, $\cos^n\theta$ in terms of sines or cosines of multiples of θ, functions in series using Taylor's and Maclaurin and Explain Hyperbolic functions, Inverse hyperbolic functions, Partial Derivatives and solve related problems.
4. Apply knowledge of partial derivatives to error and approximations, Maxima and minima and Use L'hospital Rule to evaluate indeterminate forms.
5. Relate Euler's theorems and it's corollaries to homogeneous functions and Curl, Divergence to Irrotational and Solenoidal Fields.
6. Separate given complex, hyperbolic, inverse hyperbolic and logarithmic functions into Real and Imaginary Parts.

LIST OF TUTORIALS

CO mapping with Tutorials

Tutorial No.	CO1	CO2	CO3	CO4	CO5	CO6
1	√		√			
2			√			√
3			√			√
4	√					
5			√	√		
6			√			
7			√			
8				√	√	
9				√		
10	√	√				
11		√			√	

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Tutorial 1
Complex Numbers-DeMoivre's theorem, Roots of a Complex Number

1) Find the modulus and arguments of each of the following.

(i) $\frac{1+2i}{1-(1-i)^2}$ (ii) $\frac{3-i}{2+i}+\frac{3+i}{2-i}$ (iii) $\left(\frac{2+i}{3-i}\right)^2$

2) Find two complex numbers whose sum is 4 and whose product is 8.

3) If $a+ib=\frac{1}{\alpha+i\beta}$ *prove that* $(\alpha^2+\beta^2)(a^2+b^2)=1$

4) If $\sin\alpha = i\tan\theta$ than prove that $\cos\theta+i\sin\theta=\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}$

5) If $z_1=\cos\alpha+i\sin\alpha, z_2=\cos\beta+i\sin\beta$, show that $\frac{1}{2i}\left(\frac{z_1}{z_2}-\frac{z_2}{z_1}\right)=\sin(\alpha-\beta)$.

6) If $p=\cos\theta+i\sin\theta$, $q=\cos\phi+i\sin\phi$, show that

a. $\frac{p-q}{p+q}=i\tan\left(\frac{\theta-\phi}{2}\right)$

b. $\frac{p+q}{p-q}\times\frac{pq-1}{pq+1}=\frac{\sin\theta+\sin\phi}{\sin\theta-\sin\phi}$

7) If $(1+\cos\theta+i\sin\theta)(1+\cos 2\theta+i\sin 2\theta)=u+iv$

Prove that $u^2+v^2=16\cos^2\frac{\theta}{2}\cos^2\theta$ *and* $\frac{v}{u}=\tan\frac{3\theta}{2}$

8) Prove that $\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}=\sin\theta+i\cos\theta$ Hence prove that

$$\left(1+\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}\right)^5+i\left(1+\sin\frac{\pi}{5}-i\cos\frac{\pi}{5}\right)^5=0$$

9) Prove that $\frac{(1+i)^8\left(\sqrt{3}-i\right)^4}{(1-i)^4\left(\sqrt{3}+i\right)^8}=\frac{-1}{4}$

10) Find the modulus and amplitude of $\frac{1+i\sqrt{3}}{\left(\sqrt{3}-i\right)^{17}}$.

11) Prove that $\left(1+i\sqrt{3}\right)^n+\left(1-i\sqrt{3}\right)^n=2^{n+1}\cos\left(\frac{n\pi}{3}\right)$

12) If $x_\gamma = \cos\frac{\pi}{3^\gamma} + i\sin\frac{\pi}{3^\gamma}$, Prove that $x_1 x_2 x_3 ..... = i$

13) If $x+\frac{1}{x}=2\cos\theta$, $y+\frac{1}{y}=2\cos\phi$, $z+\frac{1}{z}=2\cos\psi$ prove that

i) $xyz+\frac{1}{xyz}=2\cos(\theta+\phi+\psi)$

ii) $x^3y^3+\frac{1}{x^3y^3}=2\cos(3\theta+3\phi)$

14) If α,β are the roots of the equation $x^2-2x+2=0$ prove that $\alpha^n+\beta^n=2.2^{\frac{n}{2}}\cos\frac{n\pi}{4}$.

Hence deduce that $\alpha^8+\beta^8=32$

15) Prove that $\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n=\cos n\left(\frac{\pi}{2}-\theta\right)+i\sin n\left(\frac{\pi}{2}-\theta\right)$

16) If $\sin\alpha=i\tan\theta$, prove that $\cos\theta+i\sin\theta=\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$.

17) Show that $[(\cos\theta-\cos\varphi)+i(\sin\theta-\sin\varphi)]^n+[(\cos\theta-\cos\varphi)-i(\sin\theta-\sin\varphi)]^n$

$=2^{n+1}\sin^n\left(\frac{\theta-\varphi}{2}\right)\cos\left(\frac{n(\theta+\varphi+\pi}{2}\right)$

18) If $a=\cos 2\alpha+i\sin 2\alpha$, $b=\cos 2\beta+i\sin 2\beta$ $c=\cos 2\chi+i\sin 2\chi$, prove that

$\sqrt{\frac{ab}{c}}+\sqrt{\frac{c}{ab}}=2\cos(\alpha+\beta-\chi)$.

19) By using DeMoivre's theorem, show that $\sin\alpha+\sin 2\alpha+....+\sin 5\alpha=\frac{\sin 3\alpha\sin\left(5\alpha/2\right)}{\sin\alpha/2}$

20) If $\sin\theta+\sin\varphi=0$ *and* $\cos\theta+\cos\varphi=0$ show that

$\cos 2\theta+\cos 2\varphi=2\cos(\pi+\theta+\varphi)$ *and* $\sin 2\theta+\sin 2\varphi=2\sin(\pi+\theta+\varphi)$

21) Find the value of $(1)^{\frac{1}{5}}$ & show that these roots can be written as $1,u,u^2,u^3,u^4$. Hence prove that $(1-u)(1-u^2)(1-u^3)(1-u^4)=5$.

22) Find the continued product of the roots of $\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$

23) Solve the equation $x^7+x^4+x^3+1=0$

24) Solve the equation $x^{10}+11x^5+10=0$

25) Solve the equation $x^4+x^3+x^2+x+1=0$

26) Find the roots of the equation $z^3=i(z-1)^3$

27) Find the cube root of unity. Prove that if α & β are complex roots then $\alpha^{3n}+\beta^{3n}=2$ where n is any int.

28) Solve $x^5=1+i$ and find the continued product of the roots.

29) Find the roots common to $x^4+1=0$ & $x^6-i=0$

30) If w is a cube root of unity then prove that $(1-w)^6=-27$

31) Show that $(4n)^{th}$ power of $\frac{1+7i}{(2-i)^2}$ is equal to $(-4)^n$ where n is positive integer.

32) Prove that $\left(\frac{-1+i\sqrt{3}}{2}\right)^n+\left(\frac{-1-i\sqrt{3}}{2}\right)^n$ = -1 if n = 3k+1 or n = 3k-1

= 2 if n = 3k

33) If $\cos\alpha+2\cos\beta+3\cos\chi=\sin\alpha+2\sin\beta+3\sin\chi=0$ .Prove that

a. $\sin 3\alpha+8\sin 3\beta+27\sin 3\chi=18(\sin(\alpha+\beta+\chi))$
b. $\cos 3\alpha+8\cos 3\beta+27\cos 3\chi=18(\cos(\alpha+\beta+\chi))$

34) If $\alpha=1+i,\quad \beta=1-i\quad and\quad \cot\varphi=x+1$ prove that,

$$\frac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta}=\sin n\varphi\ .\operatorname{cosec}^n\varphi$$

35) Solve: (i) $x^5+1=0$ (ii) $x^4-x^3+x^2-x+1=0$ (iii)

(iv) $x^{10}+33x^5+32=0$ (v) $x^6-i=0$ (vi) $x^7+64x^4+x^3+64=0$

(vii) $x^7+x^4+ix^3+i=0$

36) Express $\sin^8\theta$ in a series of cos of multiples of θ.

37) Show that $\cos^5\theta\sin^3\theta=-\frac{1}{2^7}[\sin 8\theta+2\sin 6\theta-2\sin 4\theta-6\sin 2\theta]$.

38) Prove that $16\sin^5\theta=\sin 5\theta-5\sin 3\theta+10\sin\theta$.

39) Prove that $\cos^8\theta+\sin^8\theta=\frac{1}{64}[\cos 8\theta+28\cos 4\theta+35]$.

40) If $\sin^4\theta\cos^3\theta=P_1\cos\theta+P_3\cos 3\theta+P_5\cos 5\theta+P_7\cos 7\theta$ then show that $P_1+9P_3+25P_5+49P_7=0$.

41) Express $\cos^7\theta$ in a series of cos of multiples of θ.

42) Show that $2^5\sin^4\theta\cos^2\theta=\cos 6\theta-2\cos 4\theta-\cos 2\theta+2$.

43) Prove that $\cos^6\theta-\sin^6\theta=\frac{1}{16}[\cos 6\theta+15\cos 2\theta]$.

44) Prove that $\sin^7\theta\cos^3\theta = -\frac{1}{256}\left[\sin 10\theta - 4\cos 8\theta + 3\sin 6\theta + 8\sin 4\theta - 14\sin 2\theta\right]$.

45) Prove that $\frac{\sin 6\theta}{\sin 2\theta} = 16\cos^4\theta - 16\cos^2\theta + 3$.

46) Prove that $\cos 8\theta = \cos^8\theta - 28\cos^6\theta\sin^2\theta + 70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$.

47) Show that $\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$.

48) Express $\tan 3\theta$ in terms of powers of $\tan\theta$ using DeMoivre's theorem.

49) Express $\sin 7\theta$ and $\cos 7\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$.

50) Using DeMoivre's theorem, prove that

i) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$

ii) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ and $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

51) If $\sin 6\theta = a\cos^5\theta\sin\theta + b\cos^3\theta\sin^3\theta + c\cos\theta\sin^5\theta$, find the values of a, b, c.

52) Prove that $\frac{1+\cos 9A}{1+\cos A} = \left[16\cos^4 A - 8\cos^3 A - 12\cos^2 A + 4\cos A + 1\right]^2$.

Tutorial 2
Hyperbolic functions, Separation of Real and Imaginary Parts

1) Solve the following equation for real values of x, $17\cosh x + 18\sinh x = 1$.

2) Prove that $16\sinh^5 x = \sinh 5x - 5\sinh 3x + 10\sinh x$

3) If $\cos\alpha\cosh\beta = x/2$, $\sin\alpha\sinh\beta = y/2$;Show that $\sec(\alpha - i\beta) + \sec(\alpha + i\beta) = \frac{4x}{x^2 + y^2}$.

4) Prove that $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1+\sinh^2 x}}} = -\sinh^2 x$

5) If $\cosh^{-1} a + \cosh^{-1} b = \cosh^{-1} x$ then

Prove that $a\sqrt{b^2 - 1} + b\sqrt{a^2 - 1} = \sqrt{x^2 - 1}$

6) If $\log\tan x = y$, Prove that

$\cosh ny = \frac{1}{2}\left[\tan^n x + \cot^n x\right]$

$\sinh ny = \frac{1}{2}\left[\tan^n x - \cot^n x\right]$

$\cosh(n+1)y + \cosh(n-1)y = 2\cosh ny \operatorname{cosec} 2x$

$\sinh(n+1)y + \sinh(n-1)y = 2\sinh ny \operatorname{cosec} 2x$

7) If $\cosh u = \sec\theta$, Prove that

i) $\sinh u = \tan\theta$

ii) $\tanh u = \sin\theta$

iii) $u = \log\left[\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right]$

8) If $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, Prove that

i) $\cosh u = \sec\theta$

ii) $\sinh u = \tan\theta$

iii) $\tanh u = \sin\theta$

iv) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

9) Prove that $\left(\frac{1+\tanh x}{1-\tanh x}\right)^3 = \cos 6x + \sinh 6x$

10) If $\tan\frac{x}{2} = \tan\frac{u}{2}$, Prove that

i) $\sinh u = \tan x$

ii) $\cosh u = \sec x$

11) Prove that $\left(\frac{\cosh x+\sinh x}{\cosh x-\sinh x}\right)^n = \cosh 2nx + \sinh 2nx$

12) If $\sin(\alpha+i\beta) = x+iy$, then prove that $\frac{x^2}{\cosh^2\beta}+\frac{y^2}{\sinh^2\beta}=1$ and $\frac{x^2}{\sin^2\alpha}-\frac{y^2}{\cos^2\alpha}=1$

13) If $\cosh x = \sec\theta$, Prove that

i) $x = \log(\sec\theta + \tan\theta)$

ii) $\theta = \frac{\pi}{2} - 2\tan^{-1}\left(e^{-x}\right)$

14) Prove that $\cos echx + \coth x = \coth\frac{x}{2}$

15) If $\alpha + i\beta = \tanh\left(x + i\frac{\pi}{4}\right)$, prove that $\alpha^2+\beta^2=1$

16) If $\tan\left(\frac{\pi}{6}+i\alpha\right) = x+iy$, prove that $x^2+y^2+\frac{2x}{\sqrt{3}}=1$

17) If $e^z = \sin(u+iv)$ and $z = x+iy$ then prove that $2e^{2x} = \cosh 2v - \cos 2v$

Tutorial 3

Inverse Hyperbolic functions and logarithmic functions

1) Prove that $\tanh^{-1}\cos\theta = \cosh^{-1}\cos ec\theta$
2) Prove that $\sinh^{-1}\tan\theta = \log\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)$
3) Find 1) $\sin^{-1}\left(\frac{3i}{4}\right)$, 2) $\cosh^{-1}(ix)$, 3) $\sinh^{-1}(ix)$, 4) $\cos^{-1}\left(\frac{5i}{12}\right)$
4) Show that $\sin^{-1}(ix) = 2n\pi + i\log(x+\sqrt{1+x^2})$
5) Prove that $\cosh^{-1}(\sqrt{1+x^2}) = \sinh^{-1}x$
6) Prove that $\cos^{-1}(ix) = \frac{\pi}{2} - i\log(x+\sqrt{x^2+1})$
7) If $\tan z = \frac{1}{2}(1-i)$, prove that $z = \frac{1}{2}\tan^{-1}2 + \frac{i}{4}\log\left(\frac{1}{5}\right)$
8) If $\tan(\alpha+i\beta) = e^{i\theta}$, prove that $\alpha = \frac{n\pi}{2}+\frac{\pi}{4}, \beta = \frac{1}{2}\log\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$
9) Show that $\tan^{-1} i\left(\frac{x-a}{x+a}\right) = \frac{i}{2}\log\left(\frac{x}{a}\right)$
10) Prove that $\cos ech^{-1}x = \log\left[\frac{1+\sqrt{1+x^2}}{x}\right]$
11) Prove that $\sin^{-1}\left(e^{i\theta}\right) = \cos^{-1}\sqrt{\sin\theta} + i\log\left[\sqrt{\sin\theta}+\sqrt{\sin\theta+1}\right]$
12) Prove that $\tanh^{-1}x = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$
13) Show that roots of equation cosz=2 are $2n\pi + i\log\left(2+\sqrt{3}\right)$
14) If $\tan\left(\theta+i\phi\right) = \tan\alpha + i\sec\alpha$, prove that $e^{2\phi} = \cot\frac{\alpha}{2}, 2\theta = n\pi+\frac{\pi}{2}+\alpha$
15) Express in the form of a+ib

    i) $Log(-5)$, ii) $Log(3+4i)$, iii) $\log_{(1-i)}(1+i)$, iv) $\sqrt{i}^{\sqrt{i}}$

16) Show that $\cos(\log_e(i^i)) = 0$

17) Prove that $\log(1+e^{2i\theta}) = \log(2\cos\theta) + i\theta$

18) Prove that $\cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{a^2-b^2}{a^2+b^2}$

19) Show that $\tan\left\{i\log\frac{a-ib}{a+ib}\right\} = \frac{2ab}{a^2-b^2}$

20) Prove that $i\log\left(\frac{x-i}{x+i}\right) = \pi - 2\tan^{-1}x$

21) If $\log\left[\log\left(x+iy\right)\right] = p+iq$, then prove that $y = x\tan\left[\tan(q).\log\sqrt{x^2+y^2}\right]$

22) Prove that $\sin\log_e(i^{-i}) = 1$

23) Find the principal value of $i^{\log(1+i)}$

24) Show that $\log_i i = \frac{4n+1}{4m+1}$, where n & m are integers

25) Find the general value of $Log(1+i) + Log(1-i)$

26) Find the principal value of $(1+i\sqrt{3})^{(1+i\sqrt{3})}$

27) Prove that $\log(1+i\tan\alpha) = \log\sec\alpha + i\alpha$

28) $\log\left[\cos(x+iy)\right] = \frac{1}{2}\log\left(\frac{\cosh 2y + \cos 2x}{2}\right) - i\tan^{-1}(\tan x \tanh y)$

29) If $\frac{(1+i)^{(x+iy)}}{(1-i)^{(x-iy)}} = \alpha + i\beta$, find α and β

30) If $i^{i^{i^{\cdot^{\cdot^{\infty}}}}} = A+iB$,considering principal values, Prove that $\tan\left(\frac{\pi A}{2}\right) = \frac{B}{A}$ and

$A^2+B^2 = e^{-\pi B}$

31) If $\sqrt{i}^{\sqrt{i}^{\sqrt{i}^{\ldots\ldots\infty}}} = \alpha + i\beta$, then prove that $\alpha^2+\beta^2 = e^{-\beta/2}$

32) Prove that real part of the principal value of $(1+i)^{\log i}$ is $e^{-\frac{\pi^2}{8}}\cos\left(\frac{\pi}{4}\log 2\right)$

33) Prove that $\log(e^{i\alpha}+e^{i\beta})=\log\left\{2\cos\left(\frac{\alpha-\beta}{2}\right)\right\}+i\left(\frac{\alpha+\beta}{2}\right)$

34) If $(1+i\tan\alpha)^{(1+i\tan\beta)}$ is real, prove that its principal value is $(\sec\alpha)^{\sec^2\beta}$

38) Prove that the principal value of $(1+i\tan\alpha)^{-i}$ is $e^{\alpha}\left[\cos(\log\cos\alpha)+i\sin(\log\cos\alpha)\right]$

39) If $\log\sin(x+iy)=a+ib$, prove that

i) $2e^{2a}=\cosh 2y-\cos 2x$

ii) $\tan b=\cot x\tanh y$

40) If $p\log(a+ib)=(x+iy)\log m$, then prove that $\frac{y}{x}=\frac{2\tan^{-1}\left(b/a\right)}{\log(a^2+b^2)}$

41) Prove that $\log\left(\frac{1}{1-e^{i\theta}}\right)=\log\left(\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right)+i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$

Tutorial 4

Mean Value Theorems

Rolle's Theorem:

Verify Rolle's theorem for problem no 1 to 6

1. $f(x)=e^{-x}(\sin x-\cos x)$ in $\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$
2. $(2x-a)^{2/3}$ in $[0,a]$
3. $f(x)=|x|$ in $[-1,1]$
4. $f(x)=(x-k)\log x$ in $[1,k]$ k>1
5. $f(x)=x^2+2 \quad -1\le x\le 0$
   $=x+2 \quad 0\le x\le 1$
6. $f(x)=\dfrac{x^2-4x}{x+2}$ $in[0,4]$
7. Find c of the Rolle's theorem for $\log\left[\dfrac{x^2+ab}{(a+b)x}\right]$ on [a, b] , a > 0, b > 0
8. Prove that the equation $2x^3-3x^2-x+1=0$ has at least one root between 1 & 2.

9. At 7 p.m., a car is traveling at 50 miles per hour. Ten minutes later, the car has slowed to 30 miles per hour. Show that at some time between 7 and 7:10 the car's acceleration is exactly 120, in units of miles per hours squared.
10. Between two consecutive roots of f'(x)=0, there cannot be more than one root of f(x)=0.
11. Using the polynomial $x^4+x^3-2x^2-6x-4$, show that the polynomial $12x^3+9x^2-12x-18$ vanishes at some point between -1 and 2
12. Use Rolle's theorem to prove that between any two zeroes the polynomial $x^n+a_{n-1}x^{n-1}+\ldots..+a_1x+a_0$ there lies a zero of the polynomial

    $nx^{n-1}+(n-1)a_{n-1}x_{n-2}+\ldots.+a_1.$

13. If $\dfrac{a_0}{n+1}+\dfrac{a_1}{n}+\ldots\ldots..+\dfrac{a_{n-1}}{2}+a_n=0$ then prove that

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$ has at least one root between 0 & 1.

14. Use Rolle's theorem to show that the equation $ax^2 + bx = \left(\frac{a}{3} + \frac{b}{2}\right)$ has a root between 0 and 1.
15. Apply Rolle's theorem to $f(x) = \sin x\sqrt{\cos 2x}$ in $[0, \pi/4]$ and find c. Also prove that $0 < \sin x\sqrt{\cos 2x} < \sqrt{2}/4$ for $0 < x < \pi/6$
16. Two horses A and B run a race. They start and finish at the same time. Prove that at some point of time during the race they were running at the same speed.
17. If $f(x) = x(x+1)(x+2)(x+3)$, show that $f'(x)$ has three real roots.
18. Prove that a particle travelling along the path $f(t) = t^2 - 2t$, comes to rest at some point of time in the time interval $0 \le t \le 2$.

Lagrange's Mean Value theorem:

Examine the validity of condition and conclusion of Lagrange's Mean Value theorem for problem no 1 to 5

1. e^x on [0, 1]
2. $f(x) = x^{\frac{2}{3}}$ on [-2, 2]
3. $f(x) = 2x^2 - 7x + 10$ on [2, 5]
4. $|x|$ on [-2, 3]
5. $\tan^{-1} x$ on [0, 1]
6. If a, b are real numbers, show that there exists at least one real number c such that $b^3 + ab^2 + a^2b + a^3 = 4c^3, a < c < b$
7. If $f(x+h) = f(x) + f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x + \theta h)$, $0 < \theta < 1$, find θ as $x \to a$ if $f(x) = (x-a)^{5/2}$
8. Prove that $\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$, 0 < a < b < 1

   Hence deduce that $\frac{\pi}{6} + \frac{\sqrt{3}}{2} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}$
9. Show that the chord joining points x = 2, x = 3 or the curve $y = x^3$ is parallel to the tangent to the curve at $x = \sqrt{\frac{19}{3}}$
10. Apply Lagrange's Mean Value theorem to show that $1 < \frac{\sin^{-1} x}{x} < \frac{1}{\sqrt{1-x^2}}$ for $0 \le x < 1$

11. Find value of θ in Lagrange's Mean Value theorem for $f(x) = px^2 + qx + r$ for $a \le x \le a+h$

12. Apply Lagrange's Mean Value theorem to show that $0 < \frac{1}{x}\log\left(\frac{e^x - 1}{x}\right) < 1$ for x > 0

13. Find θ in Lagrange's Mean Value theorem for $f(x) = x^3$ for $0 \le x \le b$

14. Given the point at which tangent to the parabola $y = x^2$ in the interval (1, 3) is parallel to chord joining x = 1, x = 3

15. Prove that $\frac{1}{9} < \sqrt{66} - 8 < \frac{1}{8}$ without actual computations

16. A motorist drove 30 miles during a one hour trip. Show that the car's speed was equal to 30 miles per hour at least once during the trip.

Cauchy's Mean value Theorem:

1. Verify Cauchy's Mean Value theorem for $f(x) = \sin x$ and $g(x) = \cos x$ on $\left[0, \frac{\pi}{2}\right]$

2. If $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$ then prove that c of Cauchy's Mean Value theorem is Harmonic mean between a & b , a > 0, b > 0

3. If 1 < a < b, show that $\exists c$ satisfying a < c < b such that $\log\left(\frac{b}{a}\right) = \frac{b^2 - a^2}{2c^2}$

4. If $f(x)$ and $g(x)$ are differentiable functions in the interval $0 \le x \le 1$, such that $f(0) = 4, g(0) = 1, f(1) = 8$ and $g(1) = 3$. Prove that there exists $c, 0 < c < 1$ and $f'(c) = 2g'(c)$

5. If $f(x) = \sin x$ any $g(x) = \cos x$, show that c of Cauchy's Mean Value theorem is arithmetic mean between a & b.

6. Prove that $\frac{\sin b - \sin a}{e^b - e^a} = \frac{\cos c}{e^c}$

7. $f(x) = e^x; g(x) = e^{-x}$ in [a,b]. Find 'c' of Cauchy's Mean value theorem.

Tutorial 5

Series expansion and L'Hospital's rule

Taylor's Expansion:

1) Expand $f(x)=x^3+3x^2+15x-10$ in powers of (x-1) and hence find $f\left(\frac{11}{10}\right)$
2) Expand $\log x$ in powers of i) (x-1) , ii) (x-2)
3) Arrange in powers of x , by Taylor's theorem, $7+(x+2)+3(x+2)^3+(x+2)^4$
4) Expand $\tan^{-1}(x+h)$ in powers of h and hence find the value of $\tan^{-1}(1.003)$ up to 5 decimal places.
5) Using Taylor's Series, find $\sqrt{9.12}$ correct up to 5 decimal places.
6) Using Taylor's Series, find approximate value of $\sin(30^\circ, 30')$
7) Expand $\tan^{-1} x$ in powers of (x-1)
8) Expand $\sin\left(\frac{\pi}{6}+x\right)$ up to x^4
9) Using Taylor's Series, evaluate $\sqrt{1.02}$ up to 4 decimal places.
10) Expand $f(x)=x^4-3x^3+2x^2-x+1$ in powers of (x-3).
11) Expand $\tan^{-1}(x+h)$ in powers of h and hence find the value of $\tan^{-1}(1.003)$ up to 5 decimal places.

Maclaurin Series expansion

Prove the following using Maclaurin series expansion:

1) $\log\left(\frac{1+x}{1-x}\right)=2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+.....\right]$
2) $e^x\cos x=1+x-\frac{x^3}{3}+......$
3) $e^{e^x}=e\left[1+x+x^2+\frac{5}{6}x^3+.......\right]$
4) $\log\left(\frac{\tan x}{x}\right)=\frac{x^2}{3}+\frac{7}{90}x^4+......$
5) $\log\left[\log(1+x)^{1/x}\right]=-\frac{x}{2}+\frac{5x^2}{24}+\frac{x^3}{8}+....$

6) $\cos^3 x = 1 - \frac{3x^2}{2} + \frac{7x^4}{8}......$

7) $e^{e^x} = e\left[1 + x + x^2 + \frac{5}{6}x^3 + ...\right]$

8) $e^{x\sin x} = 1 + x^2 + \frac{1}{3}x^4 + \frac{1}{120}x^6 + ...$

9) $\log\left(\frac{\tan x}{x}\right) = \frac{x^2}{3} + \frac{7}{90}x^4 + ...$

Expand the following functions as Maclaurin series

1) $e^{\sin x}$
2) $\log(1 + x + x^2 + x^3)$ (up to 8th power)
3) $e^{ax}\cos bx$
4) $\sin(e^x - 1)$
5) $e^{x\cos x}$

Using Maclaurin's Series Prove the following:

1) $\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + ...$
2) $\log(\sec x + \tan x) = x + \frac{x^3}{6} + \frac{x^5}{24} + ...$
3) If $x^3 + y^3 + xy - 1 = 0$ Prove that
4) $y = 1 - \frac{x}{3} - \frac{26x^3}{81} - .....$
5) Using M.S. expand $(1 + e^x)$ in power of x up to x^4
6) $\log\sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + ...$
7) $\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} + ...$
8) If $y^3 + y - 2x = 0$ prove that $y = 2x - 8x^3 + 96x^5......$
9) Prove that $(1+x)^x = 1 + x^2 - \frac{x^3}{2} + \frac{5x^4}{6} + ...$
10) If $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + ...$ then prove that $y = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$
11) Find the Maclaurin series of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

L'Hospital's Rule:

1. $\lim_{x\to 1}\frac{1+\cos\pi x}{x^3-3x+2}$
2. $\lim_{x\to 0}\frac{x^2+2\cos x-2}{x\sin^3 x}$
3. $\lim_{x\to 0}\frac{\sin x-\tan^{-1}x}{x^2\log(1+x)}$
4. $\lim_{x\to 0}\frac{e^x-e^{x\cos x}}{x-\sin x}$
5. If $\lim_{x\to 0}\frac{ae^x-be^{-x}-cx}{x-\sin x}=4$, find a, b, c.
6. $\lim_{x\to\frac{1}{2}}\frac{\cos^2\pi x}{e^{2x}-2ex}$
7. $\lim_{x\to 0}\frac{\sin x-x+\frac{1}{6}x^3}{x^5}$
8. $\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}$
9. If $\lim_{x\to 0}\frac{\sin 3x+ax+bx^3}{x^3}=0$, find a, b
10. $\lim_{x\to 1}\frac{x^x-x}{1-x+\log x}$
11. $\lim_{x\to 1}\frac{e^x+\log\frac{1-x}{e}}{\tan x-x}$
12. $\lim_{x\to 1}\frac{\sec^2 x-2\tan x}{1+\cos 4x}$
13. $\lim_{x\to 0}\left[\frac{x(1-a\cos x)+b\sin x}{x^3}\right]=\frac{1}{3}$.Find a,b
14. $\lim_{x\to 0}\left[\frac{x(1+a\cos x)-b\sin x}{x^3}\right]=1$. Find a,b
15. Evaluate $\lim_{x\to\infty}\left[\frac{a_1^{1/x}+a_2^{1/x}+.....+a_n^{1/x}}{n}\right]^{nx}$
16. $\lim_{x\to 0}\left[\frac{a\sin^2 x+b\log\cos x}{x^4}\right]=\frac{1}{2}$.Find a,b

17. $\lim_{x\to 0}\left[\frac{ae^{x}-be^{-x}-cx}{x-\sin x}\right]=4$. Find a,b,c

18. Find $\lim_{x\to\infty}\left[\frac{1^{1/x}+2^{1/x}+2^{1/x}}{3}\right]^{3x}$ using L'Hospital rule.

19. Find $\lim_{x\to 1}\frac{\log(1-x)}{\tan\frac{\pi x}{2}}$ using L'Hospital rule.

Tutorial 6

Partial Derivatives and Composite functions

Basics

1) If $u=\cos\left(\sqrt{x}+\sqrt{y}\right)$ Prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\sin\left(\sqrt{x}+\sqrt{y}\right)=0$.

2) If $\log\left(x^2+y^2\right)$ Prove that $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$.

3) If $u=e^{xyz}$ Prove that $\frac{\partial^3 u}{\partial x\partial y\partial z}=\left(1+3xyz+x^2y^2z^2\right)e^{xyz}$.

4) If $u=\log\left(x^2+y^2+z^2\right)$ Prove that $x\frac{\partial^2 u}{\partial y\partial z}=y\frac{\partial^2 u}{\partial z\partial x}=z\frac{\partial^2 u}{\partial x\partial y}$.

5) If $z=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$ Prove that $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=\frac{x^2-y^2}{x^2+y^2}$.

6) If $z(x+y)=x^2+y^2$, prove that $\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$.

7) If $u=\log\left(x^3+y^3+z^3-3xyz\right)$ then prove that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\frac{-9}{\left(x+y+z\right)^2}$.

8) If $z=x^y+y^x$, then prove that $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$.

9) If $z=\tan(y+ax)+(y-ax)^{\frac{3}{2}}$ then prove that $\frac{\partial^2 z}{\partial x^2}=a^2\frac{\partial^2 z}{\partial y^2}$.

10) If $u=\log\left(\tan x+\tan y+\tan z\right)$, prove that $\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}=2$.

11) If $z=ct^{\frac{-1}{2}}e^{\frac{-x^2}{4a^2t}}$ then prove that $\frac{\partial z}{\partial t}=a^2\frac{\partial^2 z}{\partial x^2}$.

12) If $u=f(r)$, $r^2=x^2+y^2+z^2$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=f''(r)+\frac{2}{r}f'(r)$

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13) If $\frac{1}{u^2}=x^2+y^2+z^2$ then prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$.

14) If $u=f(r^2)$, $r^2=x^2+y^2+z^2$ then prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=4r^2f''\left(r^2\right)+6f'\left(r^2\right)$.

15) If $x = r\cos\theta$, $y = r\sin\theta$ then prove that

i) $\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}$ ii) $\frac{\partial x}{\partial \theta} = r^2 \frac{\partial \theta}{\partial x}$ iii) $\left[x\frac{\partial x}{\partial r} + y\frac{\partial y}{\partial r}\right]^2 = x^2 + y^2$

16) If $x = r\cos\theta$, $y = r\sin\theta$ then prove that $\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$.

17) If $x = r\cos\theta$, $y = r\sin\theta$ then prove that $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$.

18) If $u = Ae^{-gx}\sin(nt - gx)$ where A, g, n are constants satisfies the equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ then show that $ag = \sqrt{\frac{n}{2}}$.

19) If $z = xf(x+y) + yg(x+y)$ then prove that $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$.

20) If $u = \log\left(x^3 + y^3 - x^2 y - xy^2\right)$ then prove that

i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$

ii) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{-4}{(x+y)^2}$.

21) If $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$ and $a^2 + b^2 + c^2 = 1$ then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

22) If $u = xyf\left(\frac{y}{x}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$.

23) If $u = f(r)$, $r^2 = x^2 + y^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$.

24) If $u = r^m$, $r^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m(m+1)r^{m-2}$.

25) If $\theta = t^n e^{\frac{-r^2}{4t}}$,find n which satisfies the equation $\frac{\partial \theta}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial \theta}{\partial r})$.

Composite Functions

1) If $z = xy^2 + x^2 y, \quad x = at^2, y = 2at$ find $\frac{dz}{dt}$.

2) If $u = x^2 + y^2 + z^2, \quad x = e^{2t}, y = e^{2t}\cos 3t, z = e^{2t}\sin 3t$ find $\frac{du}{dt}$.

3) If $z = xe^y, \quad x = 2t, y = 1 - t^2$, prove that $\frac{dz}{dt} = 2e^y\left(1 - 2t^2\right)$.

4) If $z = \sin^{-1}(x - y), \quad x = 3t, y = 4t^3$, prove that $\frac{dz}{dt} = \frac{3}{\sqrt{1 - t^2}}$.

5) If $z = f(u, v), \quad u = x^2 - y^2, v = y^2 - x^2$ prove that $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$.

6) If $x^2 = au + bv, \quad y^2 = au - bv, \quad z = f(x, y)$. Prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$.

7) If $x = e^u + e^{-v}, \quad y = e^{-u} - e^v, \quad z = f(x, y)$ .Prove that $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$.

8) If $u = \log\left(x^2 + y^2\right), \quad v = \frac{y}{x}, \quad z = f(u, v)$ .Prove that $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \left(1 + v^2\right)\frac{\partial z}{\partial v}$.

9) If $u = f(x - y, y - z, z - x)$ then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

10) If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$ then prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$.

11) If $u = f\left(\frac{y - x}{xy}, \frac{z - x}{xz}\right)$ then prove that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

12) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

13) If $u = f(x, y), \quad x = r\cos\theta, \quad y = r\sin\theta$ then prove that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$.

14) If $x = e^u \sec v, \quad y = e^u \tan v, \quad z = f(x, y)$ .Prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \cos^2 v\left(\frac{\partial z}{\partial v}\right)^2\right].$$

15) If $u = e^x, \quad v = e^y, \quad z = f(x, y)$, then prove that $\frac{\partial^2 z}{\partial x \partial y} = uv\frac{\partial^2 z}{\partial u \partial v}$.

First Year

Tutorial 7

Composite functions and Implicit functions

1) If $u = lx + my$, $v = ly - mx$, $z = f(u,v)$. Prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \left(l^2 + m^2\right)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$.

2) If u=f(x,y), $x = r\cos\theta$, $y = r\sin\theta$ then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ transform into

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

OR

Transform $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ into polar coordinates.

OR

If $u = f(x, y)$, where $x = r\cos\theta,\ y = r\sin\theta$, prove that $u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$.

3) If $x = u + v$, $y = uv$ and V is function of x & y then prove that

$$\frac{\partial^2 V}{\partial u^2} - 2\frac{\partial^2 V}{\partial u \partial v} + \frac{\partial^2 V}{\partial v^2} = \left(x^2 - 4y\right)\frac{\partial^2 V}{\partial y^2} - 2\frac{\partial V}{\partial y}.$$

4) If $x = e^u \cos v$, $y = e^u \sin v$, $z = f(x, y)$ prove that

i) $x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$ ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

5) If $u = f(x^n - y^n, y^n - z^n, z^n - x^n)$ then prove that $\frac{1}{x^{n-1}}\frac{\partial u}{\partial x} + \frac{1}{y^{n-1}}\frac{\partial u}{\partial y} + \frac{1}{z^{n-1}}\frac{\partial u}{\partial z} = 0$.

6) If $u = f(x, y)$, $x = \cosh\varphi\cos\phi$, $y = \sinh\varphi\sin\phi$ then prove that

$$\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial \phi^2} = (\sinh^2\varphi + \sin^2\phi)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

7) If $z = f(x, y)$, $x = u + v,\ y = uv$ then prove that $\frac{\partial^2 z}{\partial x^2} - y\frac{\partial^2 z}{\partial y^2} = \frac{1}{u - v}\left(u\frac{\partial^2 z}{\partial u^2} - v\frac{\partial^2 z}{\partial v^2}\right)$.

8) If $x = e^v \sec u,\ y = e^v \tan u$ *and* ϕ *is function of x and y* then prove that

$$\cos u\left[\frac{\partial^2\phi}{\partial u\partial v}-\frac{\partial\phi}{\partial u}\right]=xy\left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right]+\left(x^2+y^2\right)\frac{\partial^2\phi}{\partial x\partial y}.$$

9) If $u=f\left(ax^2+2hxy+by^2\right)$ and $v=\varphi\left(ax^2+2hxy+by^2\right)$, show that $\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right)$.

10) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $xe^y+ye^z+\log x-2-3\log 2=0$

11) If $y^{x^y}=\sin x$, find $\frac{dy}{dx}$.

12) If $y^x+x^y=(x+y)^{(x+y)}$, find $\frac{dy}{dx}$.

13) If $y\log(\cos x)=x\log\sin y$, find $\frac{dy}{dx}$.

14) If $u=\sin\left(x^2+y^2\right)$ and $a^2x^2+b^2y^2=c^2$, find $\frac{du}{dx}$

15) If $x^x y^y z^z=c$, show that x=y=z, $\frac{\partial^2 z}{\partial x\partial y}=-\left[x\log ex\right]^{-1}$ and

$$\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=\frac{2\left(x^2-2\right)}{x\left(1+\log x\right)}.$$

Tutorial 8

Homogeneous functions- Euler's Theorem, Errors and Approximations

1) If $u = e^{\frac{x}{y}} \sin\left(\frac{x}{y}\right) + e^{\frac{y}{x}} \cos\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$
2) If $u = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + 2u = 0$
3) Verify Euler's theorem for $u = \frac{x + y + z}{\sqrt{x} + \sqrt{y} + \sqrt{z}}$
4) Verify Euler's theorem for the following:

   (i) $u = \sin^{-1}\left(x/y\right)$ (ii) $u = ax^2 + 2hxy + by^2$ (iii) $u = \sqrt{x} + \sqrt{y} + \sqrt{z}$

   (iv) $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ (v) $u = \frac{x\left(x^3 - y^3\right)}{x^3 + y^3}$

5) If $u = f(v)$ where v is homogeneous function of x & y of degree n, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nvf'(v)$
6) If $u = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{x + y}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.
7) If $u = x^3 e^{\frac{-x}{y}}$, find $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$
8) If $u = x^3\left(\tan^{-1}\left(\frac{y}{x}\right) + \frac{y}{x}e^{\frac{-y}{x}}\right) + y^{-3}\left(\sin^{-1}\left(\frac{x}{y}\right) + \frac{x}{y}\log\left(\frac{x}{y}\right)\right)$ then prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 9u$
9) If $u = \frac{\left(x^2 + y^2\right)^m}{2m(2m-1)} + xf\left(\frac{y}{x}\right) + \varphi\left(\frac{x}{y}\right)$, find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.
10) If $u = \frac{\left(x^3 y^3 z^3\right)}{x^3 + y^3 + z^3} + \log\left(\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

11) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

12) If $u = \sin^{-1}\left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

13) If $u = \left(\frac{x}{y}\right)^{\frac{y}{x}}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

14) Verify Euler's theorem for i) $u = x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right)$

ii) $u = 3x^2yz + 5xy^2z + 4xyz^2$.

15) If $u = x^4 y^2 \sin^{-1}\left(y/x\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 6u$.

16) If $u = \sin^{-1}\left(\frac{x^{1/3} + y^{1/3}}{x^{1/2} - y^{1/2}}\right)^{1/2}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u$.

17) If $u = \tan^{-1}\left(\frac{x + y}{x^{1/2} + y^{1/2}}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$.

18) If $u = \tan^{-1}\left(\frac{x^3 + y^3}{2x + 3y}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u$.

19) If $u = \sin^{-1}\left(\frac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0$.

20) If $u = x\, f\left(y/x\right) + g\left(y/x\right)$, show that $u = x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.

21) If $y = x\cos u$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.

22) If $u = x^4 \sin^{-1}\left(\frac{y}{x}\right) + x^6 \tan^{-1}\left(\frac{y}{x}\right)$, find the value of

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \quad \text{at } x=1,\ y=1.$$

23) If $u = \cos^{-1}\left(\frac{x^3 + y^3}{x^2 + y^2}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\cot u$.

24) If $z = \log(x^2 + y^2) + \left(\frac{x^2 + y^2}{x + y}\right) - 2\log(x + y)$, find the value of $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$.

25) If $u = e^{x^2 f\left(\frac{y}{x}\right)}$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$.

26) If $u = \sin^{-1}\left(\frac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{5}} + y^{\frac{1}{5}}}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{400}\tan u(\tan^2 u - 19)$.

27) If $3u = \log\left(\frac{x^3 + y^3}{x^2 + y^2}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-1}{3}$.

28) If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

29) If $u = \log x + \log y$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$.

30) If $u = \operatorname{cosec}^{-1}\sqrt{\left(\frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}\right)}$, then prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left[\frac{13}{12} + \frac{\tan^2 u}{12}\right].$$

31) If $u = \tan^{-1}\left(\frac{x^2 + y^2}{x - y}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -2\sin^3 u\cos u.$$

32) If $u = \sin^{-1}\left(\frac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-\sin u\cos 2u}{4\cos^3 u}$

33) If $u = \frac{x^2y^2z^2}{x^2 + y^2 + z^2} + \cos\left(\frac{xy + yz}{x^2 + y^2 + z^2}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4\frac{x^2y^2z^2}{x^2 + y^2 + z^2}$.

34) If $u = \tan^{-1}\left(\frac{y^2}{x}\right)$, show that $u = x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\sin 2u\,\sin^2 u$.

35) If $u = \frac{x^3y^3z^3}{x^3 + y^3 + z^3} + \log\left(\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 6\frac{x^3y^3z^3}{x^3 + y^3 + z^3}$.

36) If $u = \log\left(\frac{x^4 - y^4}{x - y}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -3$.

37) If $u = \tan^{-1}(x^2 + 2y^2)$ show that

i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$

ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2\sin u\cos 3u$

38) If $u=\sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/6}+y^{1/6}}\right)$ prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{144}\tan u(\tan^2 u-11)$.

39) If $u=\sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2\tan u$.

40) If $f(x,y)$ and $\phi(x,y)$ are homogeneous functions of x,y of degree p and q respectively and $u=f(x,y)+\phi(x,y)$, show that

$$f(x,y)=\frac{1}{p(p-q)}\left[x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right]-\frac{q-1}{p(p-q)}\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right].$$

41) Find the value of $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$ if $u=\sin^{-1}\left(x^3+y^3\right)^{2/5}$.

Errors and approximation

1. The period of a simple pendulum is given by $T=2\pi\sqrt{\frac{l}{g}}$. If T is computed using $l=8ft,\ g=32ft./\sec^2$, find approximate error in T if true values are $l=8.05ft,\ g=32.01ft./\sec^2$.
2. The H.P. required to propel a steamer varies as the cube of the velocity and the square of length. If there is 3% increase in velocity and 4% increase in length, find the % increase in H.P.
3. Using the formula $R=\frac{E}{I}$, find the maximum error and percentage of error in R if $I=20$ with a possible error of 0.1 and $E=120$ with a possible error of 0.05 and $R=6$.
4. Find the percentage error in calculating the area of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, when error of 1% is made in measuring its major and minor axes.
5. Find the possible percentage error in computing the parallel resistance r of three resistance r_1, r_2, r_3 from the formula $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$ if r_1, r_2, r_3 are each error by 1.2%
6. The quantity Q of water flowing over a triangular notch is given by $Q=CH^{\frac{5}{2}}$ where H is head of water and C is constant. Find the % error in Q if the error in H is that it is measured 0.198 instead of 0.2

7. The height of a cone is h=30cms and the radius of the base is r = 10cms. How will the volume of a cone change if h is increased by 3mm and r is decreased by 1mm?

8. In calculating surface area of a cylinder error by 1% each are found in measuring height and base radius. Find the % error in calculating total surface area?

9. A closed rectangular box with unequal sides a, b, c has its edges slightly altered by amount δa, δb, and δc respectively so that both its surface area and volume remained unchanged. Prove that

$$\frac{\delta a}{a^2(b-c)} = \frac{\delta b}{b^2(c-a)} = \frac{\delta c}{c^2(a-b)}$$

10. The central angle of a circular sector is 80° and the radius is 20cm. If the angle is reduced by one unit by how much should the radius of a sector be increased so that the area of sector remains unchanged.
11. Find approximate error in surface area of the rectangular parallelepiped of sides a, b, c due to error δ in measuring each side.
12. Find the % error in the area of ellipse due to 1.5% error in major and minor axes.
13. Show that the error in calculating the time period of a pendulum, $T = 2\pi\sqrt{l/g}$ is zero if an error of μ% is made in measuring its length and gravity.
14. One side of rectangle is a=10cms and other side b=24cms. How will the diagonal c of the rectangle change if a is increased by 4mm and b is decreased by 1mm.
15. If $e^z = \sec x \cos y$ and error of magnitude h & -h are made in estimating x & y where x & y are found to be $\frac{\pi}{3}$ & $\frac{\pi}{6}$ resp. Find corresponding error in z.
16. At a distance 20 meters from the foot of the tower the elevation of the top is 60° .If the possible error in measuring distance and elevation are 1cm and 1minute. Find the approximate error in the calculated height.
17. The sides of a triangle are measured as 15cms and 20cms included angle being 60°. If the sides are error by 1% and angle by 2%. Find the percentage error in determining the area of a triangle and remaining side of triangle.
18. In estimating the cost of a pile of bricks measured $2m \times 15m \times 1.2m$, the top of the pile is stretched 1% beyond the standard length. If the count is 450 bricks in 1 cubic m and bricks cost Rs. 450 per thousand, find the approximate error in the cost.
19. If $f(x,y,z) = x^2 y^3 z^{\frac{1}{10}}$, find the approximate value of f when x=1.99, y=3.01, z=0.98
20. Find $\left[(11.99)^2 + (5.01)^2\right]^{\frac{1}{2}}$ by using theory of approximation.
21. Find approximately e^{xyz} when at (0.01, 1.01, 2.01)

22. Find $\left[(2.92)^3+(5.87)^3\right]^{\frac{1}{5}}$ by using theory of approximation.

23. If $f(x,y)=\left(160-x^3-y^3\right)^{\frac{1}{3}}$, find approximate value of $f(2.1,2.9)-f(2,3)$

24. If $f(x,y,z)=x^3y^2z^4$, find approximate value of $f(1.99,3.01,0.99)$

25. Find the approximate value of $\sqrt{27}\ \sqrt[3]{1021}$.

26. Determine approximately $\sqrt{25.15}$ using the theory of approximation.

27. Find the approximate value of $\sqrt[4]{(1.9)^3+(2.1)^3}$.

28. Find the approximate value of f at the given point : $f(x,y)=\sqrt{x^2+y^2}\ at\ (3.01,4.03)$..

29. Find the approximate value of $f(x,y)=x^3+y^3$ when $x=3.025$ and $y=4.152$.

First Year

Tutorial 9
Maxima, Minima: Second derivative Test and Lagrange's Multiplier Method

1. Find extreme values of

a) $x^3 + xy^2 + 21x - 12x^2 - 2y^2$
b) $x^3 y^2 (12 - 3x - 4y)$
c) $x^3 y^2 (1 - x - y)$
d) $x^4 + y^4 + 4xy - 2x^2 - 2y^2$
e) $x^3 + y^3 - 63(x + y) + 12xy$
f) $x^2 y - 3x^2 - 2y^2 - 4y + 3$
g) $x^2 + y^2 + 6x + 12$
h) $x^4 + y^4 - x^2 - y^2 + 1$
i) $x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$
j) $2(x^2 - y^2) - x^4 + y^4$
k) $x^2 - 2xy + \frac{1}{3} y^3 - 3y$
l) $2\left(x^2 - y^2\right) - x^4 + y^4$
m) $f(x, y) = x^3 y^2 (1 - x - y)$
n) $x^4 + y^4 - 2x^2 + 4xy - 2y^2$
o) $\sin x \sin y \sin(x + y)$

2. If a rectangle box with open top is to have surface area 48sq.units. Find the dimensions of the box, so that the volume is minimum.
3. Find the maximum value of $v(x, y, z) = xyz$ subject to the constraint $2x + 2y + 2z = 108$.
4. The temperature T at any point (x, y, z) in space is $T = kxyz^2$, where k is a constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
5. The sum of the three positive numbers is 1. Determine the maximum value of their product.
6. A rectangular box open at the top is have volume of 108cm cubic. Find the dimensions of the box requiring least material.
7. Divide 24 into three parts such that the continued product of the first, square of the second and cube of the third is maximum.
8. Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.
9. The sum of three positive numbers is a. Determine the maximum value of their product.
10. Find the maximum volume of the parallelepiped inscribed in the sphere $x^2 + y^2 + z^2 = a^2$.
11. Find the maximum volume of the parallelepiped inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
12. Find the points on the surface $z^2 = xy + 1$ which are at the least distance from the origin

Lagrange's method of Multipliers with one constraint

1. Use Lagrange's Method to determine minimum distance from origin to the plane $x+2y+3z=14$.
2. Use Lagrange's Method to determine the point on the plane $x+2y+3z=13$ closest to (1,1,1)
3. Find the dimensions of the rectangular box with open top of maximum capacity and surface area of 108 sq. units, using Lagrange's Method.
4. Divide 'a' into three parts such that their product is maximum.
5. Prove that $u=8xyz$ is stationary at $x=\frac{a}{\sqrt{3}}$, $y=\frac{b}{\sqrt{3}}$, $z=\frac{c}{\sqrt{3}}$ subjected to the condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.
6. If $u=\frac{a^3}{x^2}+\frac{b^3}{y^2}+\frac{c^3}{z^2}$ where $x+y+z=1$, prove that stationary value of u is given by $x=\frac{a}{a+b+c}$, $y=\frac{b}{a+b+c}$, $z=\frac{c}{a+b+c}$.
7. Find the points on the surface $x^2+y^2+z^2=1$ which are at maximum distance from (2, 1, 3).
8. If $u=x^2+y^2+z^2$ where $\varphi\equiv ax+by+cz-p=0$, find the stationary value of u.
9. Prove that the stationary value of $x^m y^n z^p$ under the condition $x+y+z=a$ is $m^m n^n p^p\left(\frac{a}{m+n+p}\right)^{m+n+p}$.
10. If $\frac{3}{x}+\frac{4}{y}+\frac{5}{z}=6$, find the values of x, y, z such that $x+y+z$ is minimum.
11. If $u=a^3x^2+b^3y^2+c^3z^2$ where $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, show that stationary value of u is given by $x=\frac{a+b+c}{a}$, $y=\frac{a+b+c}{b}$, $z=\frac{a+b+c}{c}$.
12. Find the dimensions of a closed rectangular box with maximum volume that can be inscribed in unit sphere.
13. Find the maximum volume of the parallelepiped inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
14. Find the maximum and minimum values of $f(x,y,z)=x-2y+5z$ on sphere $x^2+y^2+z^2=30$
15. If the temperature T at any point (x, y, z) on the surface of the sphere $x^2+y^2+z^2=1$ is $T=400xyz^2$, find the highest temperature.

16. If $xyz = 8$, find the values of x, y, z for which $u = \dfrac{5xyz}{x+2y+4z}$.

Lagrange's method of Multipliers with two constraints

1. Consider extremizing the function $f(x, y, z) = xyz$ subject to the constraints $g_1(x, y, z) = x + y + z = 1$ and $g_2(x, y, z) = x + y - z = 0$.
2. Find the maximum and minimum of $f(x, y, z) = 4y - 2z$ subject to the constraints $2x - y - z = 2$ and $x^2 + y^2 = 1$.
3. Find the stationary value of $u = x^2 + y^2 + z^2$ subject to $ax^2 + by^2 + cz^2 = 1 \quad and \quad lx + my + nz = 0$.
4. Find point on sphere $x^2 + y^2 + z^2 = 1$ closest to (1, 2, 3).
5. Maximize $x + y + z$ when $x^2 + y^2 + z^2 = 1$ and $x^2 - y^2 + z = 0$.
6. Use Lagrange's Method of undetermined multipliers to find the minimum value of $u = x^2 + y^2 + z^2$ subject to the conditions $x + y + z = 1$ and $xyz = 1$
7. Prove that the stationary values of $u = \dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4}$ where $lx + my + nz = 0$ and $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ are the roots of $\dfrac{l^2a^4}{1-a^2u} + \dfrac{m^2b^4}{1-b^2u} + \dfrac{n^2c^4}{1-c^2u} = 0$

Tutorial 10

Scalar and vector product of three or four vectors, Vector differentiation

Problems on Scalar triple product and Vector triple product

1. Show that $(\vec{p}-\vec{q}).[(\vec{q}-\vec{r})\times(\vec{r}-\vec{p})]=0$
2. If $\vec{a},\vec{b},\vec{c}$ are the position vectors of the points A,B,C prove that the vector $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$ is perpendicular to the plane of the triangle ABC
3. If $\vec{l},\vec{m},\vec{n}$ are non coplanar vectors, prove that $\left[\vec{l}\ \vec{m}\ \vec{n}\right](\vec{a}\times\vec{b})=\begin{vmatrix}\vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}\\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}\\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}\end{vmatrix}$
4. Prove that $[\vec{p}.(\vec{q}\times\vec{r})]\ [\vec{a}.(\vec{b}\times\vec{c})]=\begin{vmatrix}\vec{p}.\vec{a} & \vec{p}.\vec{b} & \vec{p}.\vec{c}\\ \vec{q}.\vec{a} & \vec{q}.\vec{b} & \vec{q}.\vec{c}\\ \vec{r}.\vec{a} & \vec{r}.\vec{b} & \vec{r}.\vec{c}\end{vmatrix}$
5. Prove that the points $(2,1,1),\ (0,1,-3),\ (3,2,-1),\ (7,2,7)$ are coplanar.
6. Find the volume of the parallelepiped spanned by the vectors **a** = (- 2, 3, 1), **b** = (0, 4, 0), and **c** = (- 1, 3, 3).
7. Find the volume of the tetrahedron formed by $(1,1,3),\ (4,3,4),\ (5,2,7),\ (6,4,8)$
8. A parallelepiped has concurrent edges OA,OB,OC of lengths a,b,c along the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{3},\ \frac{x}{2}=\frac{y}{1}=\frac{z}{3},\ \frac{x}{3}=\frac{y}{1}=\frac{z}{2}$ respectively. Find the volume of the parallelepiped.
9. If $\vec{a}=3\vec{i}-2\vec{j}+2\vec{k},\vec{b}=6\vec{i}+4\vec{j}-2\vec{k},\vec{c}=3\vec{i}+2\vec{j}+4\vec{k}$ Find $\vec{a}\times(\vec{b}\times\vec{c}),(\vec{a}\times\vec{b})\times\vec{c}$. Are they same?
10. Find scalars p,q if $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\times\vec{c}$ where $\vec{a}=2\vec{i}+\vec{j}+p\vec{k},\vec{b}=\vec{i}-\vec{j},\vec{c}=4\vec{i}+q\vec{j}+2\vec{k}$
11. Prove that $[\vec{b}\times\vec{c}\ \ \vec{a}\times\vec{c}\ \ \vec{a}\times\vec{b}]\ =[\vec{a}\ \vec{b}\ \vec{c}]^2$
12. If $\vec{a},\vec{b},\vec{c}$ are coplanar vectors prove that $\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}$ are coplanar.
13. If $\vec{a},\vec{b},\vec{c}$ are non coplanar vectors prove that $\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}$ are non coplanar. Obtain scalars l,m,n, such that $\vec{a}=l(\vec{b}\times\vec{c})+m(\vec{c}\times\vec{a})+n(\vec{a}\times\vec{b})$.
14. Prove that the vectors $\vec{a}\times\left(\vec{b}\times\vec{c}\right),\vec{b}\times\left(\vec{c}\times\vec{a}\right)$ and $\vec{c}\times\left(\vec{a}\times\vec{b}\right)$ are coplanar.
15. If a vector $\vec{x}$ and a scalar λ satisfy the equation $\vec{a}\times\vec{x}=\lambda\vec{a}+\vec{b}$ and $\vec{a}.\vec{x}=1$, find the values of λ and $\vec{x}$ in terms of $\vec{a}$ and $\vec{b}$. Also determine them if $\vec{a}=\vec{i}-2\vec{j}$ and $\vec{b}=2\vec{i}+\vec{j}-2\vec{k}$
16. Show that $\left[\vec{d}\times\left(\vec{a}\times\vec{b}\right)\right]\left(\vec{a}\times\vec{c}\right)=\left[\vec{a}\,\vec{b}\,\vec{c}\right]\left(\vec{a}\cdot\vec{d}\right)$

Problems on Scalar product and Vector product of four vectors:

1. Verify Lagrange's identity for the following vectors: $\vec{a}=\vec{i}-2\vec{j}-2\vec{k}$, $\vec{b}=2\vec{i}+\vec{j}-\vec{k}$, $\vec{c}=\vec{i}+3\vec{j}-2\vec{k}$ and $\vec{d}=2\vec{i}+\vec{j}-3\vec{k}$
2. If $\vec{a},\vec{b},\vec{c}$ are non coplanar vectors , prove that
$$\left(\vec{b}\times\vec{c}\right)\times\left(\vec{a}\times\vec{d}\right)+\left(\vec{c}\times\vec{a}\right)\times\left(\vec{b}\times\vec{d}\right)+\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=-2\left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{d}$$
3. Prove that $\left[\left(\vec{a}\times\vec{b}\right)\times\left(\vec{a}\times\vec{c}\right)\right]\cdot\vec{d}=\left[\vec{a}\,\vec{b}\,\vec{c}\right]\left(\vec{a}\cdot\vec{d}\right)$
4. Prove that $\left(\vec{b}\times\vec{c}\right)\times\left(\vec{c}\times\vec{a}\right)=\left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{c}$

Problems on Vector Differentiation- curves in space, velocity, acceleration :

1. If $\frac{d\vec{a}}{dt}=\vec{u}\times\vec{a}$ and $\frac{d\vec{b}}{dt}=\vec{u}\times\vec{b}$, prove that $\frac{d}{dt}\left[\vec{a}\times\vec{b}\right]=\vec{u}\times\left(\vec{a}\times\vec{b}\right)$
2. If $\vec{r}=a\cos t\vec{i}+a\sin t\vec{j}+at\tan\alpha\vec{k}$, prove that $\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right|=a^2\sec\alpha$
3. If $\vec{r}=r\hat{r}$, prove that $\hat{r}\times\frac{d\hat{r}}{dt}=\frac{\vec{r}}{r^2}\times\frac{d\vec{r}}{dt}$
4. If $\vec{a}=\sin t\vec{i}+\cos t\vec{j}+t\vec{k},\vec{b}=\cos t\vec{i}-\sin t\vec{j}-3\vec{k},\vec{c}=2\vec{i}+3\vec{j}-\vec{k}$, find $\frac{d}{dt}\left[\vec{a}\times\left(\vec{b}\times\vec{c}\right)\right]$ at $t=0$
5. If $\overline{f}(t)$ is a unit vector, prove that $\left|\overline{f}(t)\times\frac{d\overline{f}(t)}{dt}\right|=\left|\frac{d\overline{f}(t)}{dt}\right|$
6. Find the magnitude of velocity and acceleration of a particle moving along the curve $\overline{r}(t)=\left(4\cos t,4\sin t,3t\right)$
7. A particle moving along the curve $\overline{r}(t)=\left(\vec{a}t^2,\vec{b}t,\vec{c}\right)$ where $\vec{a},\vec{b}$ and $\vec{c}$ are constant vectors. Show that the acceleration is constant.
8. Find the angle between the tangents to the curve given by $\overline{r}(t)=\left(t,t^2,t^3\right)$ at the points $t=1$ and $t=-1$.
9. Show that a particle whose position vector $\vec{r}=a\cos nt\vec{i}+b\sin nt\vec{j}$ moves in an ellipse whose centre is at the origin and that its acceleration varies directly as its distance from the centre and is directed towards it.
10. A particle moves along the curve $x=2t^2,y=t^2,z=3t-5$. Find the velocity and acceleration at $t=1$ in the direction $\vec{i}-3\vec{j}+2\vec{k}$

11. Find the tangential and normal components of acceleration at any time t of a particle whose position vector is given by $x = e^t \cos t,\ y = e^t \sin t$
12. Find the tangential and normal components of acceleration at any time t of a particle whose position vector is given by $x = \cos t + t\sin t,\ y = \sin t - t\cos t$

Tutorial 11

Gradient, Directional derivative, divergence and curl

Gradient and directional derivative

1. Find $grad\, f$ if $f = e^{xy} - x\cos\left(yz^2\right)$
2. Prove that $\nabla f\left(r\right) = f'\left(r\right)\frac{\bar{r}}{r}$ and hence prove that (a) $\nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$
3. Find $grad(\log r)$
4. Find $grad(e^{r^2})$
5. Find $f\left(r\right)$ such that $\nabla f = -\frac{\bar{r}}{r^5}$ and $f\left(1\right) = 0$
6. Find $f\left(r\right)$ such that $\nabla f = -\frac{\bar{r}}{r^5}$ and $f\left(2\right) = 3$
7. If $\nabla u = 2r^4\,\bar{r}$, find u.
8. Find directional derivative of $f = xy + yz + zx$ at (1,2,3) in the direction of $3i + 4j + 5k$
9. Find the directional derivative of $f = 2x^3y - 3y^2z$ at $P\left(1, 2, -1\right)$ in the direction towards Q(3,-1,5). In what direction from P is the directional derivative maximum? Find the magnitude of the maximum directional derivative.
10. In what direction from (3, 1,-2) is the directional derivative of $\varphi = x^2y^2z^4$ maximum? What is the rate of change at (3, 1,-2) in the direction (1, 2, 2)?
11. Find the directions in which $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$
    i. increases most rapidly at (1,1)
    ii. decreases most rapidly at (1,1)
    iii. what are the directions of zero change in f at (1,1)
12. Captain Astro is drifting in space near the sunny side of Mercury and notices that the hull of her ship is beginning to melt. The temperature in her vicinity is given by $T = e^{-x} + e^{-2y} + e^{3z}$, where x, y and z are measured in meters. If she is at $(1,1,1)$, in what direction should she proceed in order to cool fastest?

Divergence and curl

1. If $\overline{f} = (3x^2)\hat{i} + (5xy)\hat{j} + (xyz^3)\hat{k}$, find $div\,\overline{f}$ & $curl\,\overline{f}$ at (1,2,3)
2. If $\phi = x^3 + y^3 + z^3 - 3xyz$, find (a) $\overline{r} \bullet \nabla\phi$, (b) $div\overline{F}$, (c) $curl\overline{F}$; where $\nabla\phi = \overline{F}$
3. If $\overline{f} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$ prove that $\overline{f} \bullet curl\,\overline{f} = 0$
4. Prove that $\nabla\left(\nabla \bullet \frac{\overline{r}}{r}\right) = -\frac{2}{r^3}\overline{r}$
5. Prove that $\nabla \bullet \left(r\nabla \frac{1}{r^3}\right) = \frac{3}{r^4}$
6. Find $f(r)$, so that the vector $f(r)\overline{r}$ is both solenoidal and irrotational.
7. Prove that $\overline{F} = \frac{\overline{r}}{r^3}$ is both solenoidal and irrotational.
8. $\overline{F} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal. Find value of a.
9. Find a, b, c if $\overline{F} = (axy + bz^3)\hat{i} + (3x^2 - cz)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational.
10. A vector field is given by $\overline{F} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$. Prove that it is irrotational and find its scalar potential.
11. Prove that $\overline{F} = (z^2 + 2x + 3y)\hat{i} + (3x + 2y + z)\hat{j} + (y + 2zx)\hat{k}$ is irrotational and find scalar potential ϕ such that $\phi(1,1,0) = 4$.
12. If the vector $\overline{F} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2 + 2)\hat{k}$ is irrotational find its scalar potential

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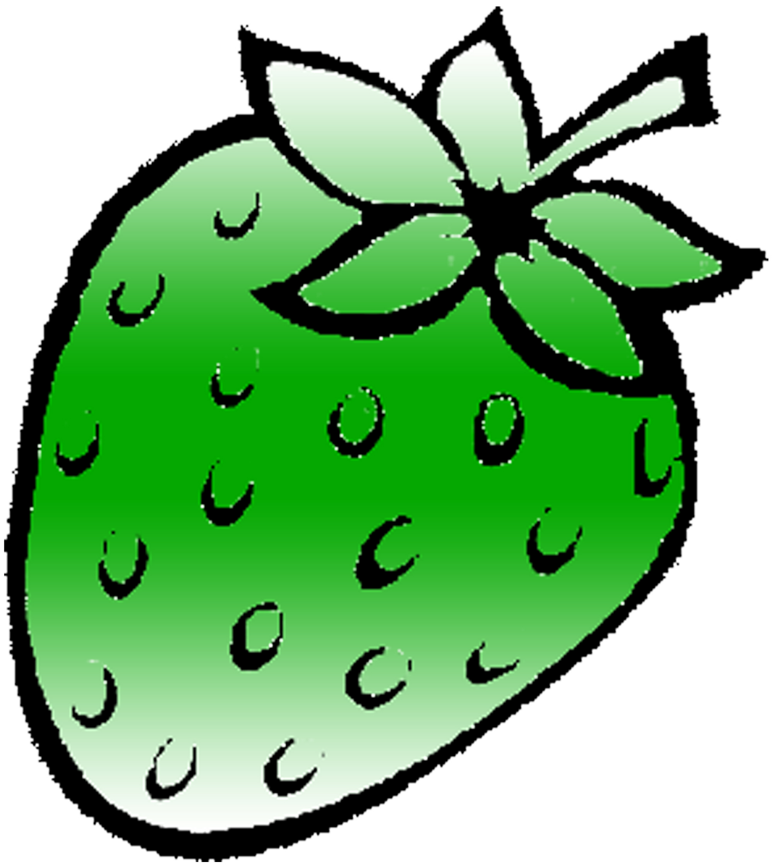
STRAWBERRY



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